Appendix 2 - Optimal control theory solution for optimal stocking rate
Investing in the commons: transient welfare creates incentives despite open access

Optimal control theory and the maximum principal provide the necessary optimal stocking rate over time that maximizes a defined objective function (Clark 1990). The optimal stocking rate is expressed as a function of shadow prices (μ_i) that are determined from constructing a Hamiltonian ($\mathcal{H}$) of the optimal control problem:

Government Hamiltonian:

$$\mathcal{H} = e^{-\rho t}(p_{RES}qE_{RES}X - c_{RES}E_{RES} + p_{ROV}qE_{ROV}X - c_{ROV}E_{ROV} - \gamma S^2) + \lambda_1(rX - bX^2 - qEX + S) + \lambda_2(\delta E_{RES}[p_{RES}qX - c_{RES}]) + \lambda_3(\delta E_{ROV}[p_{ROV}qX - c_{ROV}]) \quad \text{Eq.A2.1}$$

Lake association Hamiltonian:

$$\mathcal{H} = e^{-\rho t}(p_{RES}qE_{RES}X - c_{RES}E_{RES} - \gamma S^2) + \lambda_1(rX - bX^2 - qEX + S) + \lambda_2(\delta E_{RES}[p_{RES}qX - c_{RES}]) + \lambda_3(\delta E_{ROV}[p_{ROV}qX - c_{ROV}]) \quad \text{Eq.A2.2}$$

The current value Hamiltonian ($\widetilde{\mathcal{H}}$) equals $e^{\rho t}(\mathcal{H})$ and the current shadow price for state variable l (μ_l) equals $e^{\rho t}(\lambda_l)$. The maximum principle provides the differential equations of the current value shadow prices and the optimal stocking rate:

Current shadow prices

Lake association:

$$\dot{\mu_1} = -p_{RES}qE_{RES} - 2\gamma S(r - 2bX - q(E_{ROV} + E_{RES}) - \rho) - \delta q(\mu_2 p_{RES}E_{RES} + \mu_3 p_{ROV}E_{ROV}) \quad \text{Eq.A2.3}$$

Government:

$$\dot{\mu_1} = -q(p_{ROV}E_{ROV} + p_{RES}E_{RES}) - 2\gamma S(r - 2bX - q(E_{ROV} + E_{RES}) - \rho) - \delta q(\mu_2 p_{RES}E_{RES} + \mu_3 p_{ROV}E_{ROV}) \quad \text{Eq.A2.4}$$

$$\dot{\mu_2} = \mu_2(\rho - \delta p_{RES}qX + \delta c_{RES}) + qX(2\gamma S - p_{RES}) + c_{RES} \quad \text{Eq.A2.5}$$

Lake association:

$$\dot{\mu_3} = 2\gamma SqX - \mu_3(\delta p_{ROV}qX - \delta c_{ROV} - \rho) \quad \text{Eq.A2.6}$$

Government:

$$\dot{\mu_3} = \mu_3(\rho - \delta p_{ROV}qX + \delta c_{ROV}) + qX(2\gamma S - p_{ROV}) + c_{ROV} \quad \text{Eq.A2.7}$$

Optimal Stocking

By Equations A2.1 and A2.2, $\frac{d\widetilde{\mathcal{H}}}{dS} = -2S\gamma + \mu_1 = 0$, therefore, $\mu_1 = 2S\gamma$. Taking the derivative of both sides of this equation with respect to time and solving for $\dot{S}$ gives,

$$\dot{S} = \frac{\dot{\mu_1}}{2\gamma} \quad \text{Eq.A2.8}$$

By the Arrow principle, the necessary conditions above are sufficient if $\mathcal{H}$ evaluated at S* is concave with respect to all state variables over the planning horizon. Concavity of the Hamiltonian is determined by the properties of its Hessian matrix. However, our Hessian matrices are indeterminate, so no conclusion about the concavity of the function can be made.

Literature cited

Clark, C.W. 1990. Mathematical bioeconomics: the optimal management of renewable resources. Wiley-Interscience, Hoboken, N.J.